

Probability-based comparison of quantum states

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We address the following state comparison problem: is it possible to design an experiment enabling us to unambiguously decide (based on the observed outcome statistics) on the sameness or difference of two unknown state preparations without revealing a complete information about the states? We find that the claim “the same” can never be concluded without any doubts unless the information is complete. Moreover, we prove that a universal comparison (that perfectly distinguishes all states) also requires the complete information about the states. Nevertheless, for some measurements, the probability distribution of outcomes still allows one to make an unambiguous conclusion on the difference between the states even in the case of incomplete information. We analyze an efficiency of such a comparison of qudit states when it is based on the SWAP-measurement. For qubit states, we consider in detail the performance of special families of two-valued measurements enabling us to successfully compare at most half of the pairs of states. We conclude with a surprisingly simple example of an almost universal comparison measurement in any dimension.

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I. INTRODUCTION

The exponential scaling of the number of parameters describing multipartite quantum systems stands behind the potential power of quantum information processing. However, the same feature makes a complete characterization (tomography) of unknown quantum devices intractable. Therefore, it is of practical interest to understand which properties of physical systems require the full tomography for their determination and for which of them such a complete knowledge is redundant. In this paper we analyze the resources needed for a comparison of quantum states. Suppose given a pair of quantum systems in unknown states. The question is what experiments (if any) are capable to reveal with certainty either the difference between the states, or confirm their sameness as long as the probability distribution of measurement outcomes is identified.

By the very nature of quantum theory the events we observe in quantum experiments are random. That is, both quantum predictions and quantum conclusions are naturally formulated in terms of probabilities and uncertainty. Therefore it is surprising that there are (very specific) situations (including special instances of the comparison problem) in which individual clicks enable us to make a nontrivial unambiguous prediction, or conclusion. For example, if we are given a promise that the states are pure, then (with a nonzero probability) the difference of states can be confirmed unambiguously from a single experimental click [1, 2]. This result can be also generalized to the comparison of many pure states [3–5], the comparison of ensembles of pure states [6], and the comparison of some pure continuous-variable states [7, 8] (see also the review [9]). Unfortunately, such single-shot (non-statistical) comparison strategy fails for general mixed states [5, 10]. The reason is simple. The probability

of any outcome is strictly nonvanishing providing that a bipartite system is in the completely mixed state, for which the subsystems are in the same state. That is, for any outcome there is a situation in which the systems are the same, hence the difference cannot be concluded unambiguously. In such case any error-free conclusions need to be based on the observed probabilities of outcomes. Probability-based comparison strategies were not considered previously, so we fill this gap in the present paper.

Trivially, if the experimentally measured probabilities provide a complete information on quantum states of both systems individually, then they contain also all the information needed for the comparison. The question of our interest is whether the complete tomography is necessary. Our main goal is to design a comparison experiment providing as little redundant information as possible.

In Sec. II, we introduce the necessary mathematical notation and formulate the problem. In Sec. III, we address the existence of a universal comparison measurement. Sec. IV investigates the comparison performance of two-outcome measurements. An example of an almost universal three-outcome comparison measurement is presented in Sec. V and conclusion is the content of Sec. VI.

II. PROBLEM FORMULATION

Any quantum state is associated with the density operator $\varrho \in \mathcal{S}(\mathcal{H})$ such that $\varrho \geq 0$ and $\text{tr}[\varrho] = 1$. Hereafter, $\mathcal{S}(\mathcal{H})$ stands for the set of all states of a system associated with the Hilbert space \mathcal{H} . The statistical features of quantum measurements are fully captured by means of a positive operator-valued measure (POVM) that is a collection \mathbf{E} of positive operators (acting on \mathcal{H} and called effects) E_1, \dots, E_n summing up to the identity,

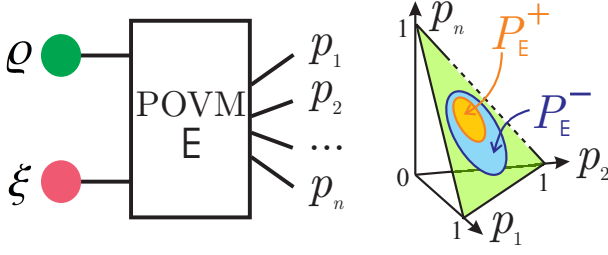


FIG. 1: Illustration of probability-based comparison. If the observed probability distribution belongs to $P_E^- \setminus P_E^+$, then states ρ and ξ are for sure different.

i.e. $\sum_{j=1}^n E_j = I$. For each state $\rho \in \mathcal{S}(\mathcal{H})$ the measurement E assigns a probability distribution $\{p_j\}_{j=1}^n \equiv \vec{p}_E$, where $p_j = \text{tr}[E_j \rho] \geq 0$ and $\sum_{j=1}^n p_j = 1$.

Let us now move on to the set of bipartite factorized states $\mathcal{S}_{\text{fac}} = \{\rho \otimes \xi : \rho, \xi \in \mathcal{S}(\mathcal{H})\} \subset \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$, where the parties ρ and ξ are the states to be compared. For a fixed measurement E we can ask how much information it reveals concerning the comparison of the subsystems.

Denote by \mathcal{S}^+ the subset of *twin-identical states*, i.e. $\mathcal{S}^+ = \{\eta \otimes \eta : \eta \in \mathcal{S}(\mathcal{H})\} \subset \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$. Similarly, let us denote by \mathcal{S}^- the subset of *non-identical states*, i.e. $\mathcal{S}^- = \{\rho \otimes \xi : \rho, \xi (\neq \rho) \in \mathcal{S}(\mathcal{H})\} \subset \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$. Obviously, $\mathcal{S}_{\text{fac}} = \mathcal{S}^+ \cup \mathcal{S}^-$. The goal of comparison is then to distinguish between sets of states \mathcal{S}^+ and \mathcal{S}^- . This goal can be achieved in our approach by considering two sets of probability distributions $P_E^\pm = \{\vec{p} : p_j = \text{tr}[E_j \omega], \omega \in \mathcal{S}^\pm\}$. In other words, since the measurement E performs the mapping $\mathcal{S}^\pm \mapsto P_E^\pm$, one can unambiguously conclude that a bipartite state ω belongs to the set \mathcal{S}^\pm if the observed probability distribution $\vec{p}_E \in P_E^\pm \setminus P_E^\mp$ (see Fig. 1).

For a fixed POVM E on $\mathcal{H} \otimes \mathcal{H}$ we may introduce the following quantities:

$$D_E(\rho \otimes \xi, \mathcal{S}^+) = \inf_{\eta \otimes \eta \in \mathcal{S}^+} \sum_j |\text{tr}[E_j(\rho \otimes \xi - \eta \otimes \eta)]|, \quad (1)$$

$$D_E(\eta \otimes \eta, \mathcal{S}^-) = \inf_{\rho \otimes \xi \in \mathcal{S}^-} \sum_j |\text{tr}[E_j(\rho \otimes \xi - \eta \otimes \eta)]|. \quad (2)$$

While $D_E(\rho \otimes \xi, \mathcal{S}^+)$ quantifies how different the states ρ and ξ are (with respect to measurement E), the value of $D_E(\eta \otimes \eta, \mathcal{S}^-)$ tells us to which extent the equivalence of twin-identical states can be confirmed.

Before we proceed further let us make one important observation: for all $\epsilon > 0$ and any state $\eta \otimes \eta$ there exists a state $\rho \otimes \xi$ such that $|\text{tr}[E(\eta \otimes \eta - \rho \otimes \xi)]| \leq \epsilon$ for any POVM effect E . In other words, in order to conclude that the states are the same no uncertainty in the specification of the probabilities $p_E(\omega) = \text{tr}[E\omega]$ is allowed. Such an infinite precision is practically not achievable, however, for our purposes we will assume the probabilities are specified exactly. The proof of the statement above is relatively straightforward. Let us set $\rho = \eta$ and $\xi = (1 - \frac{\epsilon}{2})\eta + \frac{\epsilon}{2d}I$, i.e. $\eta \otimes \eta - \rho \otimes \xi = \frac{\epsilon}{2}\eta \otimes (\eta - \frac{1}{d}I)$. Since

$$|\text{tr}[EX]| \leq \max_{E \in \mathcal{E}} \text{tr}[|EX|] \leq \text{tr}[|X|], \text{ it follows that}$$

$$|\text{tr}[E(\eta \otimes \eta - \rho \otimes \xi)]| \leq \epsilon \frac{1}{2} \text{tr}[|\eta \otimes (\eta - \frac{1}{d}I)|] \leq \epsilon. \quad (3)$$

In the last inequality we used the fact that the trace distance of states is bounded from above by one.

Our observation implies that for any measurement E we have $D_E(\eta \otimes \eta, \mathcal{S}^-) = 0$. This seems to be in contradiction with measurements which provide us with a complete information on the states of individual systems. We will refer to such measurements as *locally informationally complete* (LIC) measurements. Clearly, in case of an ideal LIC measurement the sameness can be verified. Where is the problem? Topologically, in the set of factorized states \mathcal{S}_{fac} with the trace-distance metrics, the subset \mathcal{S}^+ is closed and does not contain any interior point (therefore the distance (2) vanishes), however, it does not mean that the subset \mathcal{S}^+ is empty.

The vanishing value of the considered distance is not completely relevant if one thinks about ideal error-free experiments. In practise, the experimental noise is unavoidable, hence, from the practical point of view a conclusion on the sameness of states can never be error-free.

III. UNIVERSAL COMPARISON MEASUREMENT

We say the measurement E implements the comparison whenever $D_E(\rho \otimes \xi, \mathcal{S}^+) > 0$ for some pairs ρ, ξ . The state comparison measurement E is *universal* if $D_E(\rho \otimes \xi, \mathcal{S}^+) > 0$ for all $\rho, \xi (\neq \rho)$. This is, for instance, achieved in case of ideal LIC measurements even though the value of $D_E(\rho \otimes \xi, \mathcal{S}^+)$ can be arbitrarily small, but always strictly positive. As before, this situation is in practice not very realistic, because any error in identification of the outcome probabilities makes the conclusions (in some cases of ρ and ξ) ambiguous. However, assuming the infinite precision in specification of the probabilities, the universality can be achieved and in what follows we will assume that probabilities are identified perfectly. The potential errors can be viewed as modifications of the sets \mathcal{S}^+ and \mathcal{S}^- we are aiming to distinguish. Nevertheless, our goal is to analyze the ideal case.

Let us now demonstrate that a universal comparison can be implemented if and only if the measurement is LIC.

To start with, we are reminded that any POVM E with effects E_j linearly maps a state $\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$ into the probability vector $\vec{p} = (p_1, p_2, \dots)$, where $p_j = \text{tr}[E_j \omega]$. For LIC measurements the induced mapping $\rho \otimes \xi \mapsto \vec{p}_E$ is bijective. That proves the sufficiency. To prove the necessity let us assume the converse, i.e. suppose the measurement E is not an LIC measurement but implements a universal comparison. Since E is not LIC, the probability assignment $\rho \otimes \xi \mapsto \vec{p}_E$ is injective. Let us denote by Π^\pm and Π the images of \mathcal{S}^\pm and \mathcal{S}_{fac} under some LIC measurement, respectively, and by P_E denote the image of \mathcal{S}_{fac} under measurement E . Clearly, the

relation between $\tilde{\pi}(\varrho \otimes \xi)$ and $\tilde{p}_E(\varrho \otimes \xi)$ is linear and injective, i.e. there exist probability vectors $\tilde{\pi}_1 \in \Pi$ and $\tilde{\pi}_2 \in \Pi$ transformed into the same probability distribution $\tilde{p}_E(\varrho \otimes \xi) \in P_E$. The distributions $\tilde{\pi}_j$ transformed into the same probability vector $\tilde{p}_E(\varrho \otimes \xi)$ span a linear subspace (hyperplane) $H_{\varrho \otimes \xi}$ in the linear span of Π .

Consider now a non-pure state $\eta \otimes \eta$, then its image $\tilde{\pi}(\eta \otimes \eta)$ is an interior point of Π on the probability simplex. There exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ the neighborhood $O_\epsilon(\tilde{\pi}(\eta \otimes \eta))$ belongs to Π (on the simplex). Moreover, the intersection $O_\epsilon(\tilde{\pi}(\eta \otimes \eta)) \cap H_{\eta \otimes \eta}$ cannot be a subset of Π^+ only, because Π^+ does not contain any interior point on the simplex (if it did, the distance $D_E(\eta \otimes \eta, \mathcal{S}^-)$ would not vanish for all states η). Thus, $O_\epsilon(\tilde{\pi}(\eta \otimes \eta)) \cap H_{\eta \otimes \eta} \cap \Pi^-$ is not empty and contains points of the form $\tilde{\pi}(\tilde{\varrho} \otimes \tilde{\xi})$ such that $\tilde{\varrho} \neq \tilde{\xi}$. As both $\tilde{\pi}(\eta \otimes \eta)$ and $\tilde{\pi}(\tilde{\varrho} \otimes \tilde{\xi})$ belong to $H_{\eta \otimes \eta}$, we have $\tilde{p}_E(\tilde{\varrho} \otimes \tilde{\xi}) = \tilde{p}_E(\eta \otimes \eta)$, i.e. E is not a universal comparison measurement. This contradiction concludes the proof of the necessity.

Let us summarize two main conclusions:

- (i) In any locally informationally incomplete measurement the sameness of states cannot be confirmed.
- (ii) Universal comparison (concluding universally and unambiguously the difference of states) requires a locally informationally complete measurement.

A question that remains open is how to evaluate the overall performance of (universal, or non-universal) comparison experiments. There are several options. We can use the volume of the subset $\mathcal{S}_{\text{comp}}^-$ of states in \mathcal{S}^- that can be successfully compared, or the average value of $D_E(\varrho \otimes \xi, \mathcal{S}^+)$ with respect to some measure on the state space. In particular, these quantities read

$$|\mathcal{S}_{\text{comp}}^-|_E = \iint_{\mathcal{S}^-} \mu(d\varrho) \mu(d\xi) h(D_E(\varrho \otimes \xi, \mathcal{S}^+)), \quad (4)$$

$$\langle D_E \rangle = \iint_{\mathcal{S}^-} \mu(d\varrho) \mu(d\xi) D_E(\varrho \otimes \xi, \mathcal{S}^+), \quad (5)$$

where $h(x)$ is the Heaviside function and $\mu(d\varrho) = \mu(d\xi)$ is a measure on the state space of individual subsystems. Quite common choices for the measure μ on density operators are the ones induced by metrics, namely, by Bures distance and Hilbert–Schmidt distance (see, e.g., [12, 13] and references therein). Let us stress that $|\mathcal{S}_{\text{comp}}^-|_E = 1$ does not imply the comparison is universal, because there can be a set of measure zero for which $D_E(\varrho \otimes \xi, \mathcal{S}^+) = 0$. In such case we say that the comparison measurement is *almost universal*. It is of great interest to investigate whether there exist some almost universal comparison experiments and, in particular, how many outcomes such measurements require.

IV. TWO-VALUED COMPARISON EXPERIMENTS

Let us start our investigation with the simplest case of two-valued POVMs described by the effects $E, I - E$. In

such case

$$D_E(\varrho \otimes \xi, \mathcal{S}^+) = 2D_E(\varrho \otimes \xi, \mathcal{S}^+), \quad (6)$$

where $D_E(\varrho \otimes \xi, \mathcal{S}^+) = \min_{\eta \otimes \eta \in \mathcal{S}^+} |\text{tr} E(\varrho \otimes \xi - \eta \otimes \eta)|$. Such measurements cannot be LIC, thus, they are necessarily non-universal comparators. Nevertheless it is of practical interest to understand how good is their comparison performance.

A. SWAP-based comparison

As we have already mentioned in Sec. I, if we restrict ourselves only to pure states, then there exists a strategy to perform an unambiguous comparison (via the SWAP measurement). In such an approach, the sameness of the states cannot be concluded and this is related to the absence of the universal NOT-operation [11]. However, the strategy (if successful) can reveal the difference between the states in a single shot, hence, no collection of statistics is needed.

The key observation for such a conventional strategy is that the support of twin-identical pure states spans only the symmetric subspace of $\mathcal{H} \otimes \mathcal{H}$. Suppose projections $E_{\text{sym}}, E_{\text{asym}}$ onto symmetric and antisymmetric subspaces of $\mathcal{H} \otimes \mathcal{H}$. Since $E_{\text{sym}} + E_{\text{asym}} = I$ they form a two-valued POVM E_{SWAP} . Let us note that $E_{\text{sym}} = \frac{1}{2}(I + S)$, $E_{\text{asym}} = \frac{1}{2}(I - S)$, where S is the SWAP operator acting as $S(|\psi \otimes \varphi\rangle) = |\varphi \otimes \psi\rangle$ for all $|\psi\rangle, |\varphi\rangle \in \mathcal{H}$. It is straightforward to see that for any twin-identical pure state $|\varphi \otimes \varphi\rangle$ one has $\text{tr}[E_{\text{asym}}|\varphi \otimes \varphi\rangle\langle\varphi \otimes \varphi|] = 0$, however $\text{tr}[E_{\text{asym}}|\varphi \otimes \psi\rangle\langle\varphi \otimes \psi|] = \frac{1}{2}(1 - |\langle\varphi|\psi\rangle|^2) > 0$ if $|\psi\rangle \neq |\varphi\rangle$. Therefore, recording an outcome E_{asym} allows us to unambiguously conclude that the states are different. No statistics is needed.

Let us see how this strategy works in the case of general mixed states. A direct calculation yields

$$p_{\text{sym}} = \text{tr}[E_{\text{sym}}\varrho \otimes \xi] = \frac{1}{2}(1 + \text{tr}[\varrho\xi]), \quad (7)$$

$$p_{\text{asym}} = \text{tr}[E_{\text{asym}}\varrho \otimes \xi] = \frac{1}{2}(1 - \text{tr}[\varrho\xi]), \quad (8)$$

where we used the identity $\text{tr}[S\varrho \otimes \xi] = \text{tr}[\varrho\xi]$. The purity of states $\text{tr}[\eta^2]$ is bounded from below by $1/d$, where $d = \dim \mathcal{H}$. Thus,

$$p_{\text{sym}}(\varrho \otimes \xi) \in [\frac{1}{2}, 1) \equiv P_{\text{sym}}^-, \quad (9)$$

$$p_{\text{sym}}(\eta \otimes \eta) \in [\frac{d+1}{2d}, 1] \equiv P_{\text{sym}}^+, \quad (10)$$

where P_{sym}^\pm is the image of \mathcal{S}^\pm under the POVM effect E_{sym} . It follows that by measuring the probability $p_{\text{sym}} < (d+1)/2d$ we can with certainty conclude that the states are different. In particular, $D_{\text{SWAP}}(\varrho \otimes \xi, \mathcal{S}^+) = \max\{0, \frac{1}{d} - \text{tr}[\varrho\xi]\}$.

Suppose $\varrho = \frac{1}{d}(I + \mathbf{r} \cdot \mathbf{\Lambda})$ and $\xi = \frac{1}{d}(I + \mathbf{k} \cdot \mathbf{\Lambda})$, where $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_{d^2-1})$ is a vector formed of traceless hermitian operators Λ_j such that $\text{tr}[\Lambda_j \Lambda_k] = d\delta_{jk}$, and $\mathbf{r}, \mathbf{k} \in \mathbb{R}^{d^2-1}$ are Bloch-like vectors. Using this notation, we find that

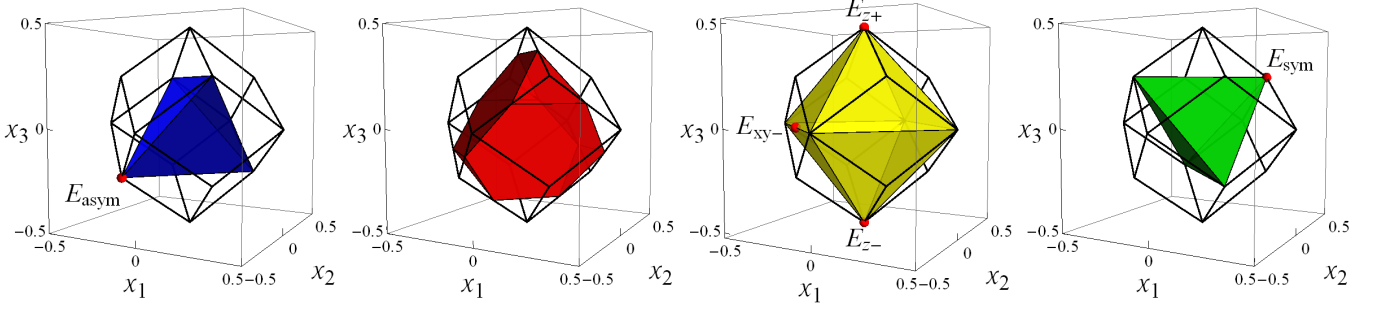


FIG. 2: Body $B(\kappa_0)$, i.e. the region of parameters $(\kappa_1, \kappa_2, \kappa_3)$ when (12) is a true POVM effect. Parameter κ_0 takes values $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, and $\frac{3}{4}$ for figures from left to right. The union $\cup_{\kappa_0 \in [0,1]} B(\kappa_0)$ is a rhombododecahedron and is depicted by solid lines. POVM effects E_{asym} , E_{sym} , $E_{z\pm}$ are vertices and POVM effects $E_{xy\pm}$ are face centers of this convex polytope.

$D_{\text{SWAP}}(\varrho \otimes \xi, \mathcal{S}^+) > 0$ only if $\mathbf{r} \cdot \mathbf{k} < 0$. That is, for each fixed $\mathbf{r} (\neq \mathbf{0})$ the set of successfully comparable Bloch-like vectors \mathbf{k} is given by an intersection of the hemisphere with the state space.

Let us stress that for qubits ($d = 2$) it is exactly the hemisphere, thus, $|\mathcal{S}_{\text{comp}}^-|_{\text{SWAP}} = \frac{1}{2}$ (see the next paragraph). In other words, a difference of states from the same hemisphere is not detected in the SWAP measurement. This implies that the approximate universality is lost.

Due to unitary invariance of the measure μ we can always treat one of the states in $D_E(\varrho \otimes \xi, \mathcal{S}^+)$ as diagonal, say ϱ . Then $\text{tr}[\varrho \xi] = \sum_{j=1}^d \varrho_{jj} \xi_{jj}$ and the integration area of $|\mathcal{S}_{\text{comp}}^-|_{\text{SWAP}}$ is split into $d!$ subsets labelled by the permutation of the labels j_1, \dots, j_d identifying the ordering $\varrho_{j_1 j_1} \geq \dots \geq \varrho_{j_d j_d}$ of eigenvalues of ϱ . The (normalized) volume of each of these subsets is $\frac{1}{d!}$. If ϱ and ξ are from mutually opposite subsets (labelled as j_1, \dots, j_d and j_d, \dots, j_1 , respectively), then $\text{tr}[\varrho \xi] = \sum_{j=1}^d \varrho_{jj} \xi_{jj} \leq \frac{1}{d}$ meaning that such pairs ϱ and ξ can be successfully compared. Therefore, we found a lower bound $|\mathcal{S}_{\text{comp}}^-|_{\text{SWAP}} \geq \frac{1}{d!}$. If ϱ, ξ are from the same subset, then $\text{tr}[\varrho \xi] = \sum_{j=1}^d \varrho_{jj} \xi_{jj} \geq \frac{1}{d}$, hence the contribution to $|\mathcal{S}_{\text{comp}}^-|_{\text{SWAP}}$ is vanishing and we can bound the fraction of comparable states also from above, $|\mathcal{S}_{\text{comp}}^-|_{\text{SWAP}} \leq 1 - \frac{1}{d!}$. For qubits both bounds coincide and give the value $|\mathcal{S}_{\text{comp}}^-|_{\text{SWAP}} = \frac{1}{2}$.

B. Qubit “diagonal” comparison experiments

In the previous section we have shown that the SWAP-based comparison enables us (in case of qubits) to successfully detect the difference for half (up to a set of zero measure) of the pairs $\varrho \otimes \xi$. Can one do better with some other two-valued measurement?

Let us stress that, for a given effect E , the states ϱ comparable with a fixed state ξ form a cut of the Bloch sphere. Therefore, if $|\mathcal{S}_{\text{comp}}^-|_{\text{SWAP}} > \frac{1}{2}$, then the com-

plete mixture must be comparable with a subset of a non-vanishing measure. Note that for the SWAP-based comparison complete mixture cannot be conclusively compared with any other state.

A general two-qubit effect takes the form

$$E = \sum_{l,m=0}^3 \varepsilon_{lm} \sigma_l \otimes \sigma_m, \quad (11)$$

where we use the notation $\sigma_0 = I$ and $\sigma_1, \sigma_2, \sigma_3$ denote the Pauli operators $\sigma_x, \sigma_y, \sigma_z$, respectively. Real coefficients ε_{lm} read $\varepsilon_{lm} = \frac{1}{4} \text{tr}[E \sigma_l \otimes \sigma_m]$ and satisfy the constraints $0 \leq E \leq I$.

For the sake of simplicity let us consider only the diagonal case, i.e. we will assume $\varepsilon_{lm} = \delta_{lm} \kappa_m$, with $\kappa_m \in \mathbb{R}$ for all $m = 0, \dots, 3$. Then

$$E_{\text{diag}} = \begin{pmatrix} \kappa_0 + \kappa_3 & 0 & 0 & \kappa_1 - \kappa_2 \\ 0 & \kappa_0 - \kappa_3 & \kappa_1 + \kappa_2 & 0 \\ 0 & \kappa_1 + \kappa_2 & \kappa_0 - \kappa_3 & 0 \\ \kappa_1 - \kappa_2 & 0 & 0 & \kappa_0 + \kappa_3 \end{pmatrix}. \quad (12)$$

Applying the positivity constraints on E of this form we obtain the following conditions:

$$0 \leq \kappa_0 - \kappa_1 - \kappa_2 - \kappa_3 \leq 1, \quad (13)$$

$$0 \leq \kappa_0 - \kappa_1 + \kappa_2 + \kappa_3 \leq 1, \quad (14)$$

$$0 \leq \kappa_0 + \kappa_1 - \kappa_2 + \kappa_3 \leq 1, \quad (15)$$

$$0 \leq \kappa_0 + \kappa_1 + \kappa_2 - \kappa_3 \leq 1. \quad (16)$$

System of inequalities (13)–(16) has a nontrivial solution whenever $0 < \kappa_0 < 1$. Indeed, if κ_0 is fixed, then each two-sided inequality of the system determines a geometric fiber between two planes in the reference frame $(\kappa_1, \kappa_2, \kappa_3)$, with the distance between the planes being equal to $\frac{1}{\sqrt{3}}$. If $0 < \kappa_0 \leq \frac{1}{4}$, then four fibers intersect to yield a tetrahedron. The intersection becomes a truncated tetrahedron if $\frac{1}{4} < \kappa_0 < \frac{1}{2}$ and finally transforms into an octahedron for the case $\kappa_0 = \frac{1}{2}$. If $\kappa_0 \geq \frac{1}{2}$ then the solution is a body obtained by the inversion $(\kappa_1, \kappa_2, \kappa_3) \rightarrow (-\kappa_1, -\kappa_2, -\kappa_3)$ of the body labelled by

parameter $(1 - \varkappa_0)$. For instance, if $\frac{3}{4} \leq \varkappa_0 < 1$, then the intersection is a tetrahedron inverted with respect to that in case $0 < \varkappa_0 \leq \frac{1}{4}$. Given \varkappa_0 we will refer to the intersection as body $B(\varkappa_0)$ (see Fig. 2).

The probabilities of the measurement outcome, corresponding to the POVM effect E_{diag} , can be readily calculated for the non-identical (different) and twin-identical (same) states and read

$$p_{\text{diff}} = p_{\text{diag}}(\varrho \otimes \xi) = \varkappa_0 + \sum_{m=1}^3 \varkappa_m r_m s_m, \quad (17)$$

$$p_{\text{same}} = p_{\text{diag}}(\eta \otimes \eta) = \varkappa_0 + \sum_{m=1}^3 \varkappa_m h_m^2, \quad (18)$$

where we used $\varrho = \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma})$, $\xi = \frac{1}{2}(I + \mathbf{s} \cdot \boldsymbol{\sigma})$ and $\eta = \frac{1}{2}(I + \mathbf{h} \cdot \boldsymbol{\sigma})$. Using the normalization constraints for Bloch vectors \mathbf{r} , \mathbf{s} , and \mathbf{h} , we obtain from Eqs. (17)–(18) that probabilities p_{diff} and p_{same} satisfy the relations

$$p_{\text{diff}} \in [\varkappa_0 - \varkappa_{\text{max}}, \varkappa_0 + \varkappa_{\text{max}}] \equiv P_{\text{diag}}^-, \quad (19)$$

$$p_{\text{same}} \in [\varkappa_0 - |\varkappa_-|, \varkappa_0 + \varkappa_+] \equiv P_{\text{diag}}^+, \quad (20)$$

respectively, where we used $\varkappa_{\text{max}} = \max\{|\varkappa_1|, |\varkappa_2|, |\varkappa_3|\}$, $\varkappa_- = \min\{0, \varkappa_1, \varkappa_2, \varkappa_3\}$, and $\varkappa_+ = \max\{0, \varkappa_1, \varkappa_2, \varkappa_3\}$. Clearly, $P_{\text{diag}}^+ \subset P_{\text{diag}}^-$, hence, there is a two-valued POVM that allows making a nontrivial conclusion on the difference of some states. Notice that the case $\varkappa_0 = \frac{3}{4}$, $\varkappa_1 = \varkappa_2 = \varkappa_3 = \frac{1}{4}$ gives the SWAP-based comparison measurement, for which whenever the measured probability p_{diag} satisfies $p_{\text{diag}} < \frac{3}{4}$, then the states ϱ and ξ are unambiguously different.

1. Fixed pair comparison

Surprisingly, there are pairs of states ϱ and ξ such that no measurement E of the form (12) can reveal their difference. A direct calculation gives that the difference for a pair of states ϱ and ξ can be concluded in the diagonal comparison experiment if

$$D_{\text{diag}}(\varrho \otimes \xi, \mathcal{S}^+) = \min_{|\mathbf{h}| \leq 1} \left| \sum_{m=1}^3 \varkappa_m (r_m k_m - h_m^2) \right| > 0. \quad (21)$$

Suppose that $r_m k_m$ is nonnegative for all m . Setting $h_m = \sqrt{r_m k_m}$ the distance (21) is vanishing for arbitrary measurement of the considered diagonal form. Let us stress that the requirement of positivity of $r_m k_m$ for all m means that signs of the Bloch vector components coincide, hence, \mathbf{k} and \mathbf{r} belong to the same octant of the Bloch ball. Let us stress, however, that the octants depend on the choice of the axes (Pauli operators), and for a given pair of states we can always fix the coordinate system in such a way that they belong to two different octants. The only exceptions are collinear vectors \mathbf{k} and $\mathbf{r} = c\mathbf{k}$ for $c \geq 0$. In fact, a pair of parallel Bloch vectors (pointing into the same direction) is indistinguishable by any diagonal measurement irrelevant of the choice of the coordinate system. In particular, it follows that none of these measurements is capable to distinguish (in the

comparison sense) the complete mixture $\varrho = \frac{1}{2}I$ from any other state, because $p_{\text{diff}}(\frac{1}{2}I \otimes \xi) = p_{\text{same}}(\frac{1}{2}I \otimes \frac{1}{2}I) = \varkappa_0$.

It is natural to ask for which pairs ϱ, ξ their difference can be identified by a suitably selected E of the considered diagonal form and whether there are some “non-diagonal” measurements enabling us to compare a pair of states containing the complete mixture.

In order to get an insight into the power of diagonal measurements, let us assume that $\varkappa_m \geq 0$, $m = 1, 2, 3$ and fix \mathbf{k} . Define a new vector $\mathbf{K} = (\varkappa_1 k_1, \varkappa_2 k_2, \varkappa_3 k_3)$. If $\mathbf{K} \cdot \mathbf{r} > 0$, then we can find \mathbf{h} such that $\mathbf{K} \cdot \mathbf{r} = \sum_{m=1}^3 \varkappa_m h_m^2$, hence $D_{\text{diag}} = 0$. If $\mathbf{K} \cdot \mathbf{r} < 0$, then $D_{\text{diag}} = |\mathbf{K} \cdot \mathbf{r}| + \sum_{m=1}^3 \varkappa_m h_m^2 > 0$ for any $\mathbf{h} \neq \mathbf{0}$. Therefore, the minimum is achieved for $\eta \otimes \eta = \frac{1}{2}I \otimes \frac{1}{2}I$ and the distance reads

$$D_{\text{diag}}(\varrho \otimes \xi, \mathcal{S}^+) = \begin{cases} 0 & \text{if } \mathbf{K} \cdot \mathbf{r} \geq 0, \\ |\mathbf{K} \cdot \mathbf{r}| & \text{otherwise.} \end{cases} \quad (22)$$

In other words, the considered diagonal measurement E enables us to verify the difference of ϱ and ξ for all ϱ satisfying the inequality $\mathbf{K} \cdot \mathbf{r} < 0$. The condition $\mathbf{K} \cdot \mathbf{r} = 0$ determines a plane containing the complete mixture (center of the Bloch ball), hence, for any measurement of the considered type and any state ξ the set of successfully comparable states ϱ is exactly a hemisphere of the Bloch ball. Fig. 3 illustrates this situation for the following choices of the diagonal measurements (POVMs):

$$\begin{aligned} E_{\text{SWAP}} &= \{E_{\text{sym}} = \frac{1}{4}(3 \cdot I \otimes I + \sum_{m=1}^3 \sigma_m \otimes \sigma_m), \\ &E_{\text{asym}} = \frac{1}{4}(I \otimes I - \sum_{m=1}^3 \sigma_m \otimes \sigma_m)\}; \end{aligned} \quad (23)$$

$$\begin{aligned} E_{xy} &= \{E_{xy+} = \frac{1}{4}(3 \cdot I \otimes I + \sum_{m=1}^2 \sigma_m \otimes \sigma_m), \\ &E_{xy-} = \frac{1}{4}(I \otimes I - \sum_{m=1}^2 \sigma_m \otimes \sigma_m)\}; \end{aligned} \quad (24)$$

$$E_z = \{E_{z\pm} = \frac{1}{2}(I \otimes I \pm \sigma_3 \otimes \sigma_3)\}. \quad (25)$$

In particular, for E_{SWAP} the “comparable hemisphere” is orthogonal to the vector \mathbf{k} . For E_{xy} the “comparable” hemisphere is orthogonal to the vector $\mathbf{k}_{\parallel} = (k_1, k_2, 0)$ being a projection of \mathbf{k} onto the xy plane. Finally, for E_z any state from the northern hemisphere is “comparable” with any state from the southern hemisphere.

It is worth noting that we have restricted ourselves to a specific form of POVM effects (12). However, even for such a simplified problem the solution looks rather sophisticated.

2. Average performance

The fact that for any given diagonal two-valued measurement the states within the same octant are not comparable means that none of them is universal neither in an approximative way. Nevertheless, it is of interest to understand which of them perform better than the others and which do not perform at all. In particular, we are interested in the answer to the following question: How many qubit states ϱ, ξ can be distinguished? As it

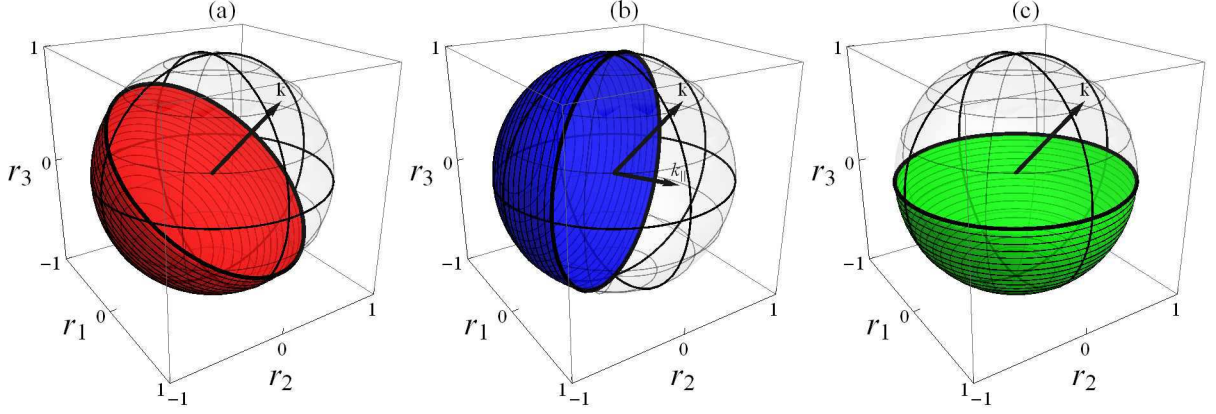


FIG. 3: States ϱ in the Bloch ball (determined by vectors \mathbf{r}) which can be distinguished from a fixed state ξ (given by vector \mathbf{k}) by using measurement \mathbf{E}_{SWAP} (a), \mathbf{E}_{xy} (b), and \mathbf{E}_z (c).

is briefly outlined in Sec. III, to answer this question it is necessary to introduce some measure μ on the state space. Once it is introduced, we can evaluate the quantities $|\mathcal{S}_{\text{comp}}^-|_{\mathbf{E}}$ (relative volume of the successfully comparable states, i.e. the comparison universality factor) and $\langle D_{\mathbf{E}} \rangle$ (average distance, i.e. the comparison quality factor).

In contrast with pure states, for density operators there exist many equivalently well-motivated measures (see, e.g., [12, 13] and references therein). We will employ two most commonly used ones, namely, the Hilbert–Schmidt measure μ_{HS} and the Bures measure μ_{B} :

$$\mu_{\text{HS}}(d\varrho) = \frac{3}{4\pi} r^2 \sin \theta dr d\theta d\varphi, \quad (26)$$

$$\mu_{\text{B}}(d\varrho) = \frac{r^2 \sin \theta}{\pi^2 \sqrt{1-r^2}} dr d\theta d\varphi, \quad (27)$$

where we used the following parametrization of Bloch vectors: $\mathbf{r} = (r \cos \varphi \sin \theta, r \sin \varphi \sin \theta, r \cos \theta)$ with $r \in [0, 1]$, $\theta \in [0, \pi]$, and $\varphi \in [0, 2\pi]$. Both these measures are spherically symmetric and the former one corresponds to the uniform coverage of the entire Bloch ball [12], i.e. the Hilbert–Schmidt measure $\mu_{\text{HS}}(T)$ of any compact set $T \subset \mathcal{S}(\mathcal{H}_2)$ equals the geometrical volume $\int_{\varrho(\mathbf{r}) \in T} d^3\mathbf{r}$ of the corresponding body inside the Bloch ball divided by $4\pi/3$. The Bures measure (27) ascribes higher weights to the states with higher purity (that are closer to the surface of the Bloch ball).

For the calculation purposes it is convenient to introduce the following (relative) density of states:

$$N_{E,\mu}^{\text{diff}}(p) = \lim_{\Delta p \rightarrow 0} \frac{1}{\Delta p} \iint_{\text{tr}[E\varrho \otimes \xi] \in [p, p+\Delta p]} \mu(d\varrho) \mu(d\xi), \quad (28)$$

whose physical meaning is that $N_{E,\mu}^{\text{diff}}(p)\Delta p$ equals the fraction of pairs $\varrho \otimes \xi$ resulting in the measurement outcome probability within the region $[p, p+\Delta p]$ for the effect E . Using the introduced function, we can readily

write

$$|\mathcal{S}_{\text{comp}}^-|_{\mathbf{E}} = \int_{p \in P_{\mathbf{E}}^- \setminus P_{\mathbf{E}}^+} N_{E,\mu}^{\text{diff}}(p) dp, \quad (29)$$

$$\langle D_{\mathbf{E}} \rangle = 2 \int_{p \in P_{\mathbf{E}}^- \setminus P_{\mathbf{E}}^+} |p - p_0| N_{E,\mu}^{\text{diff}}(p) dp, \quad (30)$$

where $P_{\mathbf{E}}^{\pm}$ stands for the image of \mathcal{S}^{\pm} under the action of POVM effect E and p_0 is simultaneously the frontier point of $P_{\mathbf{E}}^+$ and the inner point of $P_{\mathbf{E}}^-$ (if there are two such points, then $\langle D_{\mathbf{E}} \rangle$ is a sum of two integrals (30) in the corresponding regions of variable p).

In what follows we will analyze the three examples from the previous section (comparison measurements \mathbf{E}_{SWAP} , \mathbf{E}_{xy} , and \mathbf{E}_z) and compare their performance. The associated densities are depicted in Fig. 4 and explicitly written in the Appendix. We focus on these POVMs because they represent three different types of boundary (extremal in case of \mathbf{E}_{SWAP} and \mathbf{E}_z) points of the set of diagonal measurements (see Fig. 2). Other diagonal measurements will exhibit intermediate behavior with respect to these three.

We have numerically tested that for diagonal measurements the relative volume of the set of comparable states $|\mathcal{S}_{\text{comp}}^+|_{\mathbf{E}} \leq \frac{1}{2}$ irrelevant of the measure used. The considered three examples saturate this value, i.e. $|\mathcal{S}_{\text{comp}}^+|_{\mathbf{E}_{\text{SWAP}}, xy, z} = \frac{1}{2}$. In fact, there exist many measurements (of the considered family) for which the comparable set is of this size and the question is whether there are some interesting differences in their performance. As a figure of merit for this purposes we employ the average distance $\langle D_{\mathbf{E}} \rangle$, which is closely related to the quality of the fixed-pair comparison and partially quantifies also the difference of states.

The performance of three comparison measurements \mathbf{E}_{SWAP} , \mathbf{E}_{xy} , and \mathbf{E}_z is compared in Table I. It is clear that whichever measure μ is used, the mean value $\langle D_{\mathbf{E}} \rangle_{\mu}$ is greater for the measurements \mathbf{E}_{SWAP} and \mathbf{E}_z

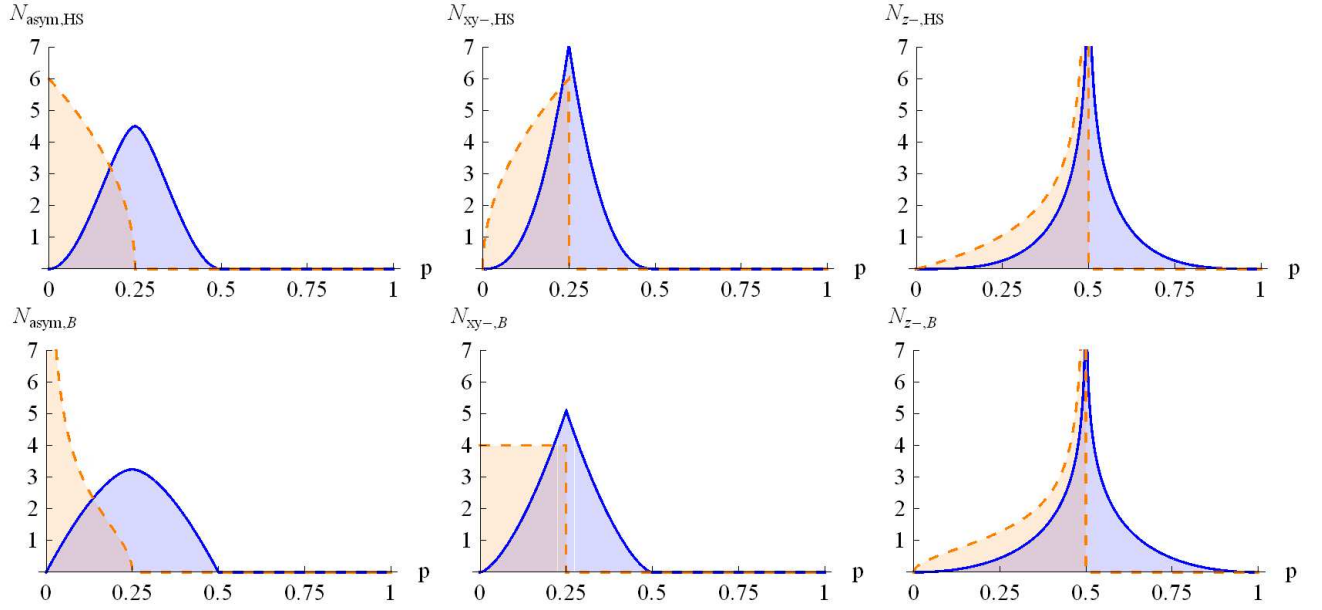


FIG. 4: Densities of states $N_{E,\mu}^{\text{diff}}(p)$ (solid) and $N_{E,\mu}^{\text{same}}(p)$ (dashed) for different POVM effects: E_{asym} (left column), E_{xy-} (middle column), and E_{z-} (right column); and for different measures: the Hilbert-Schmidt measure (top row) and the Bures measure (bottom row).

TABLE I: Effectiveness of the probability-based comparison based on different POVM effects. Mean values $\langle \cdot \rangle_\mu$ and dispersions $D_\mu[\cdot]$ of the distance $D_E(\varrho \otimes \xi, S^+)$ for different kinds of measure μ .

	E_{SWAP}	E_{xy}	E_z
$\langle D_E \rangle_{\text{HS}}$	0.07032	0.05522	0.07032
$\langle D_E \rangle_{\text{B}}$	0.09006	0.07074	0.09006
$D_{\text{HS}}[D_E]$	0.01006	0.00695	0.01506
$D_{\text{B}}[D_E]$	0.01533	0.01062	0.02314
$\sqrt{D_{\text{HS}}[D_E]} / \langle D_E \rangle_{\text{HS}}$	1.426	1.510	1.745
$\sqrt{D_{\text{B}}[D_E]} / \langle D_E \rangle_{\text{B}}$	1.375	1.457	1.689

than that for the measurement E_{xy} . Furthermore, although both POVMs E_{SWAP} and E_z lead to the same expectation values $\langle D_E \rangle_\mu$, the former one gives rise to less dispersion $D_\mu[D_E]$ and relative standard deviation $\sqrt{D_\mu[D_E]} / \langle D_E \rangle_\mu$. In addition, from Fig. 4 it follows that the effect E_{asym} results in the smallest density of states in the vicinity of the point p_0 . Such feature is very demanding because the values close to p_0 are the most affected by potential statistical errors, which are unavoidable in practise.

Using the mentioned figures of merit, we can draw a conclusion that the measurement E_{SWAP} performs (on average) the best among the considered measurements. There is yet another fact in favor for this. Fig. 4 contains also the densities of the same states $N_{E,\mu}^{\text{same}}(p)$ defined by

$$N_{E,\mu}^{\text{same}}(p) = \lim_{\Delta p \rightarrow 0} \frac{1}{\Delta p} \int_{\text{tr}[E\eta \otimes \eta] \in [p; p+\Delta p]} \mu(d\eta). \quad (31)$$

The value of the quantity $N_{E,\mu}^{\text{same}}(p)\Delta p$ tells us the number of states of the form $\eta \otimes \eta$ for which the probability $\text{tr}[E\eta \otimes \eta]$ belongs to the region $[p; p+\Delta p]$ (explicit formulas for involved effects E are given in the Appendix). One can clearly see from Fig. 4 that for the SWAP-based comparison (unlike the other two) the distribution is concentrated far from the border point p_0 . It is evident that if the measured experimentally probability p satisfies $p \in P_{\text{asym}}^+$ then one cannot judge on states being the same or different. However, even in this case it is possible to extract an additional information. In fact, the measured probability p sets a limit on the maximum trace distance between the states ϱ and ξ (providing they are different), because $\text{tr}|\varrho - \xi| \leq 2\sqrt{2p}$. Consequently, the smaller the measured probability p the closer the states ϱ and ξ are.

C. “Non-diagonal” qubit measurements

We have argued that “diagonal” qubit measurements are not able to decide on the difference of states defined by codirectional Bloch vectors, in particular, the states $\frac{1}{2}I \otimes \varrho$ are not in the comparable sets of any measurement from this family. We addressed the question whether this feature is general. In other words, whether there exists a two-valued qubit measurement allowing us to decide on the difference between the complete mixture and some other state.

Consider a qubit state ξ and its spectral decomposition $\xi = \sum_{i=1,2} \Xi_i |\Xi_i\rangle \langle \Xi_i|$ with eigenvalues $\Xi_1 \geq \Xi_2$. Suppose a two-valued measurement E with the effects

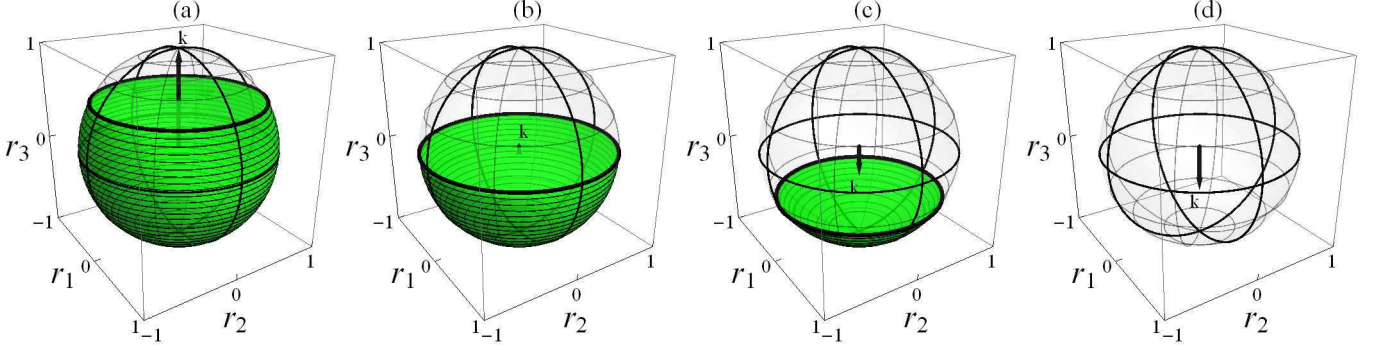


FIG. 5: States ρ inside the Bloch ball (determined by vectors \mathbf{r}) which can be distinguished from a particular fixed state $\xi = \text{diag}(\Xi_1, \Xi_2)$ (given by vector $\mathbf{k} = (0, 0, \Xi_1 - \Xi_2)$) by using the non-diagonal POVM effect $E = |\Xi_2 \otimes \Xi_1\rangle\langle \Xi_2 \otimes \Xi_1| \equiv \text{diag}(0, 0, 1, 0)$ for the following cases: $\xi = \text{diag}(1, 0)$ (a), $\xi = \text{diag}(\frac{1}{2}, \frac{1}{2})$ (b), $\xi = \text{diag}(\frac{1}{3}, \frac{2}{3})$ (c), and $\xi = \text{diag}(\frac{1}{4}, \frac{3}{4})$ (d).

$E_1 = |\Xi_2 \otimes \Xi_1\rangle\langle \Xi_2 \otimes \Xi_1|$ and $E_2 = I - E_1$. Then we have

$$\begin{aligned} D_E(\rho \otimes \xi, \mathcal{S}^+) &= 2 \inf_{\eta \otimes \eta \in \mathcal{S}^+} |\rho_{22}\Xi_1 - \eta_{11}\eta_{22}| \\ &= \begin{cases} 2\rho_{22}\Xi_1 - \frac{1}{2} & \text{if } \rho_{22} > (4\Xi_1)^{-1}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (32)$$

where $\rho_{ii} = \langle \Xi_i | \rho | \Xi_i \rangle$ and $\eta_{ii} = \langle \Xi_i | \eta | \Xi_i \rangle$, $i = 1, 2$. If, for instance, $\Xi_1 = 1$ (i.e. ξ is the north pole of the Bloch ball), then states ρ with $\rho_{22} < 1/4$ form the comparable set for ξ . They lie below latitude 60° North and include also the maximally mixed state (see Fig. 5a). If ξ is the maximally mixed state ($\Xi_1 = \Xi_2 = \frac{1}{2}$), then it is unambiguously distinguished from any state from the southern hemisphere of the Bloch ball (Fig. 5b). Applying unitary transformations of the form $U \rho U^\dagger$ and $(U \otimes U)E_{1,2}(U^\dagger \otimes U^\dagger)$, we can draw a conclusion that maximally mixed state can be effectively compared with any other qubit state, which answers our question in a positive way.

We find that on average the fraction of comparable states is smaller than $\frac{1}{2}$. In particular, $|\mathcal{S}_{\text{comp}}^-|_{\text{HS}} = \frac{3}{8}(6 - 7 \ln 2) = 0.43$ and $|\mathcal{S}_{\text{comp}}^-|_{\text{B}} = 0.42$, which means that the measurements E_{SWAP} , E_{xy} , and E_z outperform the non-diagonal measurement in this parameter. The average distance $\langle D_E \rangle$ reads values 0.1342 and 0.1524, respectively. We can see that the quality factor $\langle D_E \rangle$ is increased by the expense of the lower universality factor $|\mathcal{S}_{\text{comp}}^-|$.

Surprisingly, for a fixed pure state $\xi = |\Xi\rangle\langle \Xi|$ it is possible to design a two-outcome measurement E_Ξ such that $D_\Xi(\rho \otimes \xi, \mathcal{S}^+) > 0$ for all states $\rho \neq \xi$. In fact, suppose a POVM E_Ξ with effects $E_{\Xi 1} = \text{diag}(\frac{1}{4}, \frac{3}{8}, \frac{1}{8}, \frac{5}{8})$ and $E_{\Xi 2} = I - E_{\Xi 1}$ specified in the orthonormal basis $\{|\Xi \otimes \Xi\rangle, |\Xi \otimes \Xi_\perp\rangle, |\Xi_\perp \otimes \Xi\rangle, |\Xi_\perp \otimes \Xi_\perp\rangle\}$. Then, $p_{\Xi 1}(\rho \otimes \xi) = \frac{1}{8}(2 - \rho_{22})$ and $p_{\Xi 1}(\eta \otimes \eta) = \frac{1}{8}(2 + 3\eta_{22}^2)$, where $\rho_{22} = \langle \Xi_\perp | \rho | \Xi_\perp \rangle$ and $\eta_{22} = \langle \Xi_\perp | \eta | \Xi_\perp \rangle$. Hence $P_{\Xi 1}^- = [\frac{1}{8}, \frac{1}{4}]$ and $P_{\Xi 1}^+ = [\frac{1}{4}, \frac{5}{8}]$. Therefore, if $\rho \neq \xi$, then we necessarily observe a probability $p_{\Xi 1}$ outside the interval $P_{\Xi 1}^+$, which unambiguously identifies the difference of ρ and ξ . Such

a two-outcome experiment can be used to check whether a copy ρ of the etalon pure state ξ was really produced. Nonetheless, in spite of the seeming effectiveness of this measurement, its average performance is quite low. To be precise, the fraction of comparable states $\rho \otimes \xi$ on average reads $|\mathcal{S}_{\text{comp}}^-|_{\text{HS}} = 0.097$, or $|\mathcal{S}_{\text{comp}}^-|_{\text{B}} = 0.131$, and the quality factor is $\langle D_\Xi \rangle_{\text{HS}} = 0.0049$, or $\langle D_\Xi \rangle_{\text{B}} = 0.0079$.

V. ALMOST UNIVERSAL COMPARISON MEASUREMENT

We have shown in Sec. III that any universal comparison measurement is necessarily an LIC measurement. However, the question of an existence of an almost universal comparison measurement (which is not LIC) remains open. The maximal fraction of pairs of qubit states ρ, ξ which can be distinguished unambiguously by means of a two-outcome POVM seems to be $\frac{1}{2}$. In what follows we will give an example of a three-valued POVM, for which this fraction equals 1. This makes such a three-outcome measurement almost universal.

Consider a measurement $E_{3\text{-out}}$ composed of effects $E_1 = \text{diag}(0, 1, 0, 0)$, $E_2 = \text{diag}(0, 0, 1, 0)$, and $E_3 = \text{diag}(1, 0, 0, 1)$ in some Hilbert space basis $|j \otimes k\rangle$ of two qubits. Assuming the states are the same, the probabilities of outcomes satisfy $p_1^{\text{same}} = p_2^{\text{same}} \leq 1/4$, $p_3^{\text{same}} = 1 - p_1^{\text{same}} - p_2^{\text{same}} \geq 1/2$. In other words, the set of twin-identical states \mathcal{S}^+ is mapped onto a line inside the probability simplex (see Fig. 6). On the other hand, the elements of \mathcal{S}^- , $\rho \otimes \xi$, give rise to probability vectors \vec{p}^{diff} intersecting the line of twin-identical states, $P_{3\text{-out}}^+$, if and only if density matrices ρ and ξ have the same diagonal elements. However, the subset of the states $\rho \otimes \xi$ satisfying this peculiar requirement has zero measure in \mathcal{S}^- . Hence, almost all pairs of states ρ and ξ can be compared by the described three-outcome measurement $E_{3\text{-out}}$ (Fig. 6).

Let us note that the considered example of 3-outcome POVM is nothing else but a coarse-graining of a local

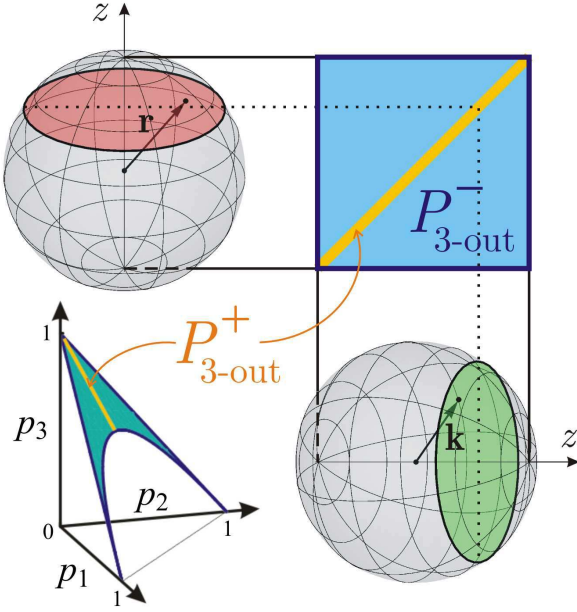


FIG. 6: Almost universal comparison measurement (for qubits) with three outcomes. States ϱ and ξ can be distinguished whenever their Bloch vectors satisfy $r_z \neq k_z$.

projective measurement applied on each of the system independently. In particular, $E_1 = E_{01}$, $E_2 = E_{10}$, and $E_3 = E_{00} + E_{11}$, where $E_{jk} = |j \otimes k\rangle\langle j \otimes k|$ ($j, k = 0, 1$) are the effects forming the local (factorized) projective measurement $\mathbf{E}_{4\text{-out}}$. In other words, if one performs the same (along the same direction) Stern-Gerlach experiment on both spin- $\frac{1}{2}$ systems, then the resulting four-outcome POVM $\mathbf{E}_{4\text{-out}}$ performs an almost universal comparison. So does the POVM $\mathbf{E}_{3\text{-out}}$, where the outcomes ‘00’ and ‘11’ are unified into a single one.

For a given state ξ the set of comparable states ϱ equals the whole Bloch ball except for a circle of states satisfying $r_z = k_z$, where $r_z = \text{tr}[\varrho\sigma_z]$ and $k_z = \text{tr}[\xi\sigma_z]$ are z -components of the corresponding Bloch vectors \mathbf{r} and \mathbf{k} , respectively. Here, we assume that σ_z is the local measurement performed on each of the states. A direct calculation of the distance (1) yields

$$D_{3\text{-out}}(\varrho \otimes \xi, \mathcal{S}^+) = \frac{1}{2} \begin{cases} |r_z - k_z| & \text{if } r_z k_z \geq 0, \\ |r_z - k_z| + |r_z k_z| & \text{otherwise.} \end{cases} \quad (33)$$

It is not hard to see that the calculation of $|\mathcal{S}_{\text{comp}}^-|$ by formula (4) results in 1, i.e. the comparison measurement $\mathbf{E}_{3\text{-out}}$ is almost universal indeed. The average distance (5) is $\langle D_{3\text{-out}} \rangle_{\text{HS}} = 0.29$, or $\langle D_{3\text{-out}} \rangle_{\text{B}} = 0.33$, i.e. substantially greater than for 2-valued measurements (cf. Sec. IV).

This qubit example can be straightforwardly generalized to any dimension. Suppose a d -dimensional system and let \mathbf{Q} be a projective measurement associated with effects $Q_j = |\psi_j\rangle\langle\psi_j|$, where $\{|\psi_j\rangle\}_{j=1}^d$ form an orthonormal basis in \mathcal{H} . Suppose the same measurement is performed on both d -dimensional systems and define a

coarse-grained POVM \mathbf{E}_{cg} with effects $E_0 = \sum_{j=1}^d Q_j \otimes Q_j$ and $E_{jk} = Q_j \otimes Q_k$ if $j \neq k$. If the states are the same, then $p_{jk}^{\text{same}} = p_{kj}^{\text{same}}$ and $p_0^{\text{same}} = 1 - 2 \sum_{j < k} p_{jk}^{\text{same}}$. Clearly, $\vec{p}^{\text{diff}} \in P_{\text{cg}}^+$ if $\langle\psi_j|\varrho|\psi_j\rangle = \langle\psi_j|\xi|\psi_j\rangle$ for all $j = 1, \dots, d$. Thus, P_{cg}^+ is a less-parametric subset of P_{cg}^- and, hence, its measure is zero. In conclusion, the measurement \mathbf{E}_{cg} consisting of $1 + d(d-1)$ effects $\{E_0, E_{jk}\}$ is an example of the almost universal comparison measurement for d -dimensional quantum systems.

VI. SUMMARY

In its essence the comparison is a binary decision problem, which we believe plays a very important role in our everyday lives. In this paper we addressed its quantum version, namely, a comparison of a pair of unknown sources of generally mixed states. We designed a new comparison strategy based on the observed statistics (not individual outcomes) of a particular comparison measurement device. It turns out that, basing upon the observed probabilities of measurement outcomes, one can sometimes draw an unambiguous conclusion on the difference between the states. In fact, it seems that a vast majority of measurements are capable to compare some pairs of states. However, as we have shown in this paper, the universal comparison of two arbitrary states requires locally informationally complete measurements, hence, a complete tomography of both sources is necessary and sufficient in order to perfectly distinguish between twin-identical states $\eta \otimes \eta$ and non-identical states $\varrho \otimes \xi$ ($\varrho \neq \xi$).

Further we analyzed the comparison performance of two-valued qubit measurements. We defined the family of “diagonal” measurements including the so-called SWAP-based comparison measurement, which is known to be useful for the single-shot unambiguous comparison of pure states. We have shown that none of these diagonal measurements is able to decide on the difference of a completely mixed state from any other mixed state. Consequently, the fraction of comparable states, $|\mathcal{S}_{\text{comp}}^-|_{\mathbf{E}}$, is at most $\frac{1}{2}$ for diagonal measurements. We compared in detail the average performance of three diagonal measurements \mathbf{E}_{SWAP} , \mathbf{E}_{xy} , and \mathbf{E}_z that are boundary (extremal in case of \mathbf{E}_{SWAP} and \mathbf{E}_z) points of diagonal measurements. Although for all of these measurements $|\mathcal{S}_{\text{comp}}^-|_{\mathbf{E}} = \frac{1}{2}$, we found differences in the distribution of distances $D_{\mathbf{E}}(\varrho \otimes \xi, \mathcal{S}^+)$. Using these considerations, we concluded that the SWAP-based comparison performs (on average) better than the other two examples.

We also provided non-diagonal comparison measurements enabling us to decide on the difference between an arbitrary state ξ and the complete mixture $\frac{1}{2}I$ or between a pure state ξ and any other state (in both cases the measurement depends on ξ). In this sense, for a given ξ the measurements of this kind overcome the performance of any diagonal measurement. However, their average performance over the set of all states result in $|\mathcal{S}_{\text{comp}}^-|_{\mathbf{E}} < \frac{1}{2}$. Despite this shortcoming, any pair of qubit states can

be compared in a suitable non-diagonal two-valued measurement.

In the remaining part we gave an example of the almost universal comparison measurement in any dimension, i.e. a measurement for which the size of the comparable set is maximal, $|\mathcal{S}_{\text{comp}}^-|_E = 1$, but still there are some pairs of states (forming a subset of measure zero), for which their difference cannot be certified. This measurement has $d(d-1)+1$ outcomes (3 in case of qubits), hence, it is not locally informationally complete, which requires the specification of $2(d^2-1)$ independent parameters (6 in case of qubits) and, consequently, the measurement with at least $2d^2-1$ outcomes (7 in case of qubits). Moreover, this measurement is experimentally very feasible to implement, because it is just a coarse-graining of a local measurement where both systems are measured by the same (d -valued) projective measurement (e.g., the Stern-Gerlach apparatus oriented along z -direction in the case of spin particles).

In summary, we have shown that the universal comparison of states (in the considered settings) is not possible, but still there exist simple almost universal comparators. In particular, for qubits the two-outcome measurements are not sufficient for almost universality, but three-outcome ones are already sufficient. A nice feature of the proposed almost universal comparison is its experimental simplicity. We left many open questions, especially concerning an optimality of the almost universal comparison measurements. As concerns the universal comparison, an optimality is directly related to the optimal complete tomography.

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Appendix A: Densities of states

The densities of states $N_{E,\mu}^{\text{diff}}(p)$ and $N_{E,\mu}^{\text{same}}(p)$ are introduced in Eqs. (28) and (31), respectively. It is worth noting that the domain of functions $N_{E,\mu}^{\text{diff}}(p)$ and $N_{E,\mu}^{\text{same}}(p)$ is P_E^- and P_E^+ , respectively. In this section, we present the

explicit formulas of these densities for the effects E_{asym} , E_{xy-} , and E_{z-} specified in Eqs. (23)–(25). We calculate the densities by using either the Hilbert–Schmidt measure (26) or the Bures measure (27) and the obtained densities are depicted in Fig. 4.

As far as POVM effect E_{asym} is concerned, $P_{\text{asym}}^- = (0, \frac{1}{2}]$ and $P_{\text{asym}}^+ = [0, \frac{1}{4}]$, consequently $p_0 = \frac{1}{4}$ and the density of states $N_{\text{asym},\mu}^{\text{diff}}(p)$ is symmetrical with respect to the point p_0 . The density of states can be calculated explicitly in the corresponding domains for the Hilbert–Schmidt measure and expressed in quadratures for the Bures measure, namely,

$$\begin{aligned} N_{\text{asym,HS}}^{\text{diff}}(p) &= \frac{9}{2} \left[1 + (4p-1)^2 (2 \ln |4p-1| - 1) \right], \\ N_{\text{asym,B}}^{\text{diff}}(p) &= \frac{32}{\pi^2} \int_{|4p-1|}^1 \sqrt{\frac{r^2 - (4p-1)^2}{1-r^2}} dr, \\ N_{\text{asym,HS}}^{\text{same}}(p) &= 6\sqrt{1-4p}, \\ N_{\text{asym,B}}^{\text{same}}(p) &= 4\sqrt{1-4p}/\pi\sqrt{p}. \end{aligned}$$

Similarly, for the effect E_{xy-} we have $P_{xy-}^- = (0, \frac{1}{2}]$, $P_{xy-}^+ = [0, \frac{1}{4}]$, and $p_0 = \frac{1}{2}$ but the densities of states differ from those obtained above and in the corresponding domains they read:

$$\begin{aligned} N_{xy-,HS}^{\text{diff}}(p) &= \frac{9}{2} \left[(1 + 2(4p-1)^2) \arccos |4p-1| - 3|4p-1| \sqrt{1-(4p-1)^2} \right], \\ N_{xy-,B}^{\text{diff}}(p) &= \frac{16}{\pi^2} \int_{\arcsin |4p-1|}^{\pi - \arcsin |4p-1|} d\theta \\ &\quad \times \int_{\frac{|4p-1|}{\sin \theta}}^1 \sqrt{\frac{r^2 \sin^2 \theta - (4p-1)^2}{(1-r^2) \sin^2 \theta}} dr, \\ N_{xy-,HS}^{\text{same}}(p) &= 12\sqrt{p}, \\ N_{xy-,B}^{\text{same}}(p) &= 4. \end{aligned}$$

Finally, the effect E_{z-} is characterized by regions $P_{z-}^- = (0, 1]$, $P_{z-}^+ = [0, \frac{1}{2}]$, $p_0 = \frac{1}{2}$ and gives rise to the following density of states:

$$\begin{aligned} N_{z-,HS}^{\text{diff}}(p) &= \frac{9}{4} \left[(1 + (2p-1)^2) (1 - \ln |2p-1|) - 2 \right], \\ N_{z-,B}^{\text{diff}}(p) &= \frac{16}{\pi^2} \int_{|2p-1|}^1 \frac{\sqrt{(r_z^2 - (2p-1)^2)(1-r_z^2)}}{r_z^2} dr_z, \\ N_{z-,HS}^{\text{same}}(p) &= 3p/\sqrt{1-2p}, \\ N_{z-,B}^{\text{same}}(p) &= 4\sqrt{2p}/\pi\sqrt{1-2p}. \end{aligned}$$

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